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SOPHUS LIE'S TRANFORMATION GROUPS.

A SERIES OF ELEMENTARY, EXPOSITORY ARTICLES.

By EDGAR ODELL LOVETT, Princeton University.

I.

1. Without entering unnecessarily into definitions which will occur more properly later, the following paragraph may serve for purposes of orientation. Among the most important notions of modern pure mathematics are the idea of a *group* and its associated notions *transformation*, *substitution*, *invariant* and *differential invariant*. Groups fall naturally and historically into two classes, *discontinuous* and *continuous*. The former are usually called *substitution groups* and are not infrequently referred to as GALOIS' groups ; the latter are known as *continuous transformation groups* and may with propriety be called LIE groups. Substitution groups find their greatest usefulness in the theory of algebraic equations, with a limited range of application to geometry ; transformation groups play a similar rôle in the theory of differential equations, with a wide application to geometry and mechanics. The idea of a substitution group in its modern signification and in its relation to the theory of algebraic equations is due to GALOIS ; LIE, after having modified and extended the idea of a substitution group, introduced the new notion into the domain of analysis and geometry and thus created his theory of transformation groups.

The great fruitfulness and remarkable simplicity of LIE's theories are their most striking characteristics. Because of their manifold applications, a thorough and systematic study of the fundamental properties of continuous groups is cer-

tain to yield the reader a liberal education in mathematics ; in addition to a knowledge of the technicalities of the theory of groups, there is gained at the same time a properly proportioned perspective of the many fields of the science brought into one domain. The group idea is a unifying principle which tends to reduce the various and in many cases apparently heterogeneous subjects of mathematics into a homogeneous body of doctrine.

It is the purpose of these notes to present some of the more elementary theorems of LIE's theories and to call attention to a few of the many applications to geometry and differential equations. The material has been drawn from the numerous published treatises* and memoirs of LIE and from his lectures delivered at the University of Leipsic in 1895 and '96. In order to an intelligent perusal of the sequel no more is required than a familiarity with the facts and processes of elementary mathematics including the simpler operations of the differential and integral calculus.

2. The simplest LIE groups are those of one parameter ; however, before proceeding to the fundamental theorems of the theory of groups of one parameter a few examples already familiar to the reader of analytical geometry as transformations of coördinates will be useful in introducing the notions.

For the sake of simplicity let the study be made in the plane. Consider the plane as a manifoldness of points, *i. e.* as a space whose space element is a point. Since it takes two independent coördinates to fix the position of a point in the plane we may say that there are ∞^2 points† in the plane or, what amounts to the same thing, that the plane is a two-dimensional space if the point is its space element. Consider the ensemble of all points of the plane and suppose that this aggregate be moved a given distance in a given direction. By this *translation* every point in the plane will be carried into the position of one of the others. In order to represent this analytically, let us suppose that the x-axis of a Cartesian coördinate system lies in the direction of the translation and that the distance through which all the points of the plane are moved is a, then the point (x, y) is carried over into the point

$$x_1 = x + a, \; y_1 = y.$$

The segment a can be given all values from $-\infty$ to $+\infty$, and if a be varied in this manner we obtain ∞^1 translations in one and the same direction or in its opposite direction.

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*A list of these treatises is to be found in the June (1897) number of the Bulletin of the American Mathematical Society or in Teubner's catalogue. The reader who desires to prosecute the study of the subject further than the scope of these notes will find the following order of attack on Lie's published works the most satisfactory: 1° Lectures on Differential Equations with Known Infinitesimal Transformations; 2° Lectures on Continuous Groups; 3° Geometry of Contact Transformations; 4° Theory of Transformation Groups, the three volumes of this treatise in their order.

†This notation is very convenient. Its general form is—If a configuration depends on n independent parameters, of which none is superfluous, the configuration assumes ∞^n positions if the parameters are allowed to vary from $-\infty$ to $+\infty$. So, for example, there are ∞^1 points on a line, ∞^2 in the plane, ∞^3 in space, since the position of the point depends on one, two, or three parameters, respectively. Similarly there are ∞^3 circles in the plane, ∞^4 straight lines in space, ∞^5 conics in the plane, and so on. The symbol ∞^n in this connection is read "n-ply infinite number of."

Suppose now that two of these translations be carried out in succession, the first through the distance a changes the point (x, y) into the point

$$x_1 = x + a, \quad y_1 = y,$$

and the second through the distance a_1 carries the new point (x_1, y_1) into the position

$$x_2 = x_1 + a_1, \quad y_2 = y_1,$$

which together with (x, y) and (x_1, y_1) lies on a parallel to the x-axis. Now it is clear geometrically, that the passage from the initial position (x, y) to the final position (x_2, y_2) can be effected by a single translation through the distance $a + a_1$, and in fact simultaneously for all points of the plane. This also appears analytically from the fact that the elimination of the intermediate position (x_1, y_1) from the above equation gives

$$x_2 = x + a + a_1, \quad y_2 = y.$$

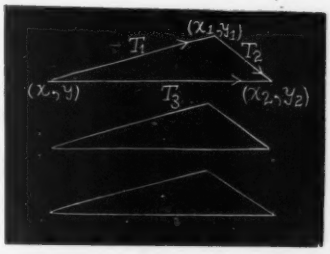
This very simple result may be formulated in the following manner:

The successive application of two translations of the family of ∞^1 *translations*

$$x_1 = x + a, \quad y_1 = y$$

is equivalent to a translation belonging to the same family.

For this reason the family is called a *group of translations*. It contains one arbitrary parameter a and hence ∞^1 translations; accordingly it is said to be a *one-parameter group*.



3. So far the translations have been limited in direction; let us now consider all translations in the plane. As in the preceding case let all the points of the plane be moved through the same distance a and in the same direction α; if a and α be given all possible values we obtain a family of ∞^2 translations which includes the preceding family as a particular case. Any one of the translations of the family changes the point (x, y) into the point

$$x_1 = x + a, \quad y_1 = y + b,$$

where a and b are two arbitrary values but remain the same for all points of the plane. If a translation, T_1, carry the point (x, y) into the position of the point (x_1, y_1), and a second translation, T_2, carry the point (x_1, y_1) over to (x_2, y_2), it is clear geometrically that the point (x, y) could have been carried directly to the position (x_2, y_2) by a single translation, T_3. The length and direction of

this third translation T_3, equivalent to the successive application of T_1 and T_2, is found by constructing the third side of the triangle formed by the translations T_1 and T_2, or in kinematical parlance, by taking the geometric sum of T_1 and T_2. This result appears analytically by eliminating x_1, y_1 from the equations representing the translations

$$T_1, \quad x_1 = x + a, \; y_1 = y + b ;$$

$$T_2, \quad x_2 = x_1 + a_1, \; y_2 = y_1 + b_1 ;$$

this elimination yields the equation

$$T_3, \quad x_2 = x + a + a_1, \; y_2 = y + b + b_1,$$

which is of the same form as the equations representing T_1 and T_2 and hence belongs to the same family as T_1 and T_2 ; therefore we conclude that

The successive application of any two translations of the family of all translations of the plane

$$x_1 = x + a, \; y_1 = y + b$$

is equivalent to a single translation belonging to the same family.

Because of the possession of this remarkable property* the family of all translations of the plane is called a *group of translations.* The *group* contains two arbitrary constants a and b, *i. e.* it has ∞^2 different translations ; for this reason the group of all translations in the plane is called a *two-parameter* group.

4. In order to present simple concrete examples illustrative of several other fundamental notions let us return to the family of all translations parallel to the x-axis

$$x_1 = x + a, \; y_1 = y ; \tag{1}$$

among these ∞^1 translations there is one to be noted, namely that one for which

*It is easy to see that this property of the equivalence of the successive applications of any two transformations of a family of transformations to a third transformation belonging to the same family is a remarkable one, peculiar to certain families, and not common to all. For example, the equations,

$$x_1 = a - x, \quad y_1 = y,$$

represent a family of ∞^1 transformations, which may be readily constructed geometrically, but a transformation S_1 changing (x, y) into

$$x_1 = a - x, \quad y_1 = y,$$

followed by S_2, changing (x_1, y_1) into

$$x_2 = a_1 - x_1, \quad y_2 = y,$$

produces, by the elimination of (x_1, y_1) from these equations, the equations

$$x_2 = (a_1 - a) + x, \quad y_2 = y,$$

which represents the transformation S_3 equivalent to the successive application of S_1 and S_2. But the x of the original family is equal to a constant minus the old x, while in S_2 the new x is equal to a constant plus the old x, hence S_3 does not belong to the same family as S_1 and S_2. The ∞^1 transformations represented by the above equation then do not form a LIE group.

$a=0$, *i. e.* a translation through the distance zero. By this translation all points of the plane remain at rest, and strictly speaking the term translation is no longer allowable. If, for the sake of continuity, the term translation is to be retained as applicable to this case also, then the translation by which every point is changed into itself is called the *identical translation*. It is to be further remarked that for every translation of this group there is a translation of the group which, carried out after the former, cancels its effect. Thus the successive application of the translations corresponding to $+a$ and to $-a$ respectively is equivalent to the translation $a-a=0$, that is, to the identical transformation. For this reason the two translations are said to be inverse.

If we put a equal to an infinitely small constant ∂t, we obtain an *infinitesimal translation*, which gives to all points of the plane only an infinitely small motion

$$x_1=x+\partial t,\ y_1=y.$$

By this translation the coördinates x, y receive infinitely small increments

$$\partial x=\partial t,\ \partial y=0,$$

and if the *infinitesimal translation* be carried out n times successively, the point (x, y) is changed into

$$x_1=x+n\partial t,\ y_1=y\ ;$$

if the infinitesimal translation be repeated an infinite number of times, or, what comes to the same thing, if n becomes infinite, then $n\partial t$ is equal to some finite quantity a and we have again a finite translation

$$x_1=x+a,\ y_1=y.$$

We shall find later on that a one-parameter group contains but one infinitesimal transformation.

Suppose that we operate on a definite point (x_0, y_0) with all translations of the one-parameter group (1) ; the point will take ∞^1 different positions :

$$x=x_0+a,\ y=y_0,$$

the aggregate of which is a parallel to the x-axis. This line, the locus of all the points into which a definite point is changed by operating on it with all the translations of the group, is called the *path curve of the point*, or *path curve of the one-parameter group*. There are altogether ∞^1 path curves of the group (1) consisting of the family of straight lines parallel to the x-axis.

Any translation of the group carries any one of the path curves, as a whole, forward in its own direction a distance a ; *i. e.* the path curve as a whole remains at rest. The path curves are invariant by all the translations of the

group. In addition to the line at infinity and the path curves, there is no other *invariant curve* by this one-parameter group, *i. e.* no other curve all of whose points are changed into points of the same curve by all the transformations of the group.

If a function of (x, y), $F(x, y)$ is to be invariant by the group

$$x_1 = x + a, \quad y_1 = y,$$

we must have $F(x_1, y_1) \equiv F(x+a, y) = F(x, y)$, for all values of a. In order to determine the function F we need only to take an infinitesimal value for a, and carry out the infinitesimal translation $x_1 = x + \partial t$, $y_1 = y$.* Taylor's series gives

$$F(x, y) + \frac{a}{1}\frac{\partial F(x, y)}{\partial x} + \frac{a^2}{1.2}\frac{\partial^2 F(x, y)}{\partial x^2} + \ldots\ldots = F(x, y),$$

or cancelling $F(x, y)$ from each side and neglecting terms of the second order

$$\frac{\partial F(x, y)}{\partial x} = 0,$$

that is, F does not contain x and is a function of y alone. Hence every function $F(y)$ is an *invariant function* by the one-parameter group (1). An *invariant function* equated to a constant gives an *invariant equation*, which represents one or more *path curves* of the group.

The reader may find it interesting to verify the group property for the following families and to determine the path curves and forms of invariant functions :

1° Rotations about a fixed point $\begin{cases} x_1 = x\cos\alpha - y\sin\alpha, \\ y_1 = x\sin\alpha + y\cos\alpha ; \end{cases}$

2° The affine transformations $x_1 = ma, \ y_1 = y ;$

3° The perspective transformations $x_1 = ax, \ y_1 = ay ;$

4° The transformations $x_1 = ax, \ y_1 = y/a ;$

5° The group of all Euclidean motions in ordinary space

$$x_1 = a_1 x + a_2 y + a_3 z + a_0,$$

$$y_1 = b_1 x + b_2 y + b_3 z + b_0,$$

$$z_1 = c_1 x + c_2 y + c_3 z + c_0.$$

The University of Chicago, 10 September, 1897.

[To be Continued.]

*In order that a function, equation or curve be invariant by all of the finite transformations of a one-parameter group, it is necessary and sufficient that the function, equation or curve be invariant by the infinitesimal transformation of the group. This theorem will be proved in the sequel.

ON A SOLUTION OF THE GENERAL BIQUADRATIC EQUATION.

By A. C. BURNHAM, Professor of Mathematics, University of Illinois, Urbana, Illinois.

Very often in mathematical work does one wish to write out without waste of time the value of the unknown in a given biquadratic equation. Nowhere in text-books or mathematical writings do I find the solution to a biquadratic given in such form that one by merely substituting in a formula may get the roots. I have found the formula here given convenient and I do not know that the formula or this particular method of getting the result has ever before been published.

Let the general biquadratic be

$$x^4+a_1x^3+a_2x^2+a_3x+a_4=0,$$

and let the roots be a, b, c, d. Then follow, as is well known,

$$\begin{aligned} a+b+c+d&=-a_1,\\ ab+ac+ad+bc+bd+cd&=a_2,\\ abc+abd+acd+bcd&=-a_3,\\ abcd&=a_4. \end{aligned}$$

Now let

$$\left.\begin{aligned} z_1&=ab+cd\\ z_2&=ac+bd\\ z_3&=ad+bc \end{aligned}\right\} \quad \text{.....I.}$$

Then it follows that,

$$\begin{aligned} z_1+z_2+z_3&=a_2,\\ z_1z_2+z_1z_3+z_2z_3&=(ab+cd)(ac+bd)+(\ \ )(\ \ )+(\ \ )(\ \ )\\ &=a^2bc+ab^2d+c^2ad+cbd^2+\ldots\ldots+\ldots\ldots\\ &=\Sigma a^2bc=a_1a_3-4a_4, \end{aligned}$$

and

$$\begin{aligned} z_1z_2z_3&=(ab+cd)(ac+bd)(ad+bc)\\ &=\Sigma a^3bcd+\Sigma a^2b^2c^2\\ &=a_3^2+a_1^2a_4-4a_2a_4. \end{aligned}$$

Then z_1, z_2, z_3 are therefore the roots of the reducing cubic:

$$z^3-a_2z^2+(a_1a_3-4a_4)z-(a_3^2+a_1^2a_4-4a_2a_4)=0 \quad \text{..... II.}$$

Now from I we have

$$z_1^2 = a^2b^2 + c^2d^2 + 2abcd$$
$$= a^2b^2 + c^2d^2 + 2a_4,$$
$$\therefore z_1^2 - 4a_4 = a^2b^2 + c^2d^2 - 2abcd = (ab - cd)^2,$$
$$\therefore \sqrt{z_1^2 - 4a_4} = ab - cd \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (a),$$
$$\text{but } z_1 = ab + cd \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (b).$$

Therefore by adding (a) and (b),

$$ab = \tfrac{1}{2}\{z_1 + \sqrt{z_1^2 - 4a_4}\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (c),$$

and by subtracting (a) from (b) we have

$$cd = \tfrac{1}{2}(z_1 - \sqrt{z_1^2 - 4a_4}) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (d).$$

In the same manner we get

$$ac = \tfrac{1}{2}(z_2 + \sqrt{z_2^2 - 4a_4}) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (e),$$
$$bd = \tfrac{1}{2}(z_2 - \sqrt{z_2^2 - 4a_4}) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (f),$$
$$ad = \tfrac{1}{2}(z_3 + \sqrt{z_3^2 - 4a_4}) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (g),$$
$$bc = \tfrac{1}{2}(z_3 - \sqrt{z_3^2 - 4a_4}) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (h).$$

But $ab + ac + ad = a(b + c + d)$
$$= (-a_1 - a)a, \text{ since } b + c + d = -a_1 - a$$
$$= -a^2 - a_1 a.$$

Also $ab + ac + ad = \tfrac{1}{2}\{z_1 + z_2 + z_3 + \sqrt{z_1^2 - 4a_4} + \sqrt{z_2^2 - 4a_4} + \sqrt{z_3^2 - 4a_4}$
from (c), (e), and (g). Therefore,

$$a^2 + a_1 a + \tfrac{1}{2}\{a_2 + \sqrt{z_1^2 - 4a_4} + \sqrt{z_2^2 - 4a_4} + \sqrt{z_3^2 - 4a_4}\} = 0,$$

which is a biquadratic equation giving the value of one root a, *i. e.*

$$a = \frac{-a_1 \pm \sqrt{a_2 - 2\{a_2 + \sqrt{z_1^2 - 4a_4} + \sqrt{z_2^2 - 4a_4} + \sqrt{z_3^2 - 4a_4}\}}}{2} \quad \ldots\ldots$$

The four roots are, therefore,

$$\left.\begin{matrix} a \\ b \\ c \\ d \end{matrix}\right\} = \tfrac{1}{2}\left\{-a_1 \pm \sqrt{a_1^2 - 2\{a_2 \pm \sqrt{z_1^2 - 4a_4} \pm \sqrt{z_2^2 - 4a_4} \pm \sqrt{z_3^2 - 4a_4}} \right. \quad \ldots\ldots \text{III},$$

where the sequence of signs under the main radical, as can be seen from formulæ (c) to (h), is

$$\begin{array}{lccc} \text{for } a, & + & + & + \\ \text{for } b, & + & - & - \\ \text{for } c, & - & + & - \\ \text{for } d, & - & - & + \end{array}$$

For the z_1, z_2, z_3 in this solution III must be substituted the roots of the cubic II.

EXAMPLE. As an example take the biquadratic

$$x^4-x^3-7x^2+x+6=0.$$

Here we have,

$$a_1=-1, \quad a_3=1,$$
$$a_2=-7, \quad a_4=6,$$

from which the cubic becomes $z^3+7z^2-25z-175=0$, of which the roots are 5, -7, and -5. Thus the roots of the biquadratic are

$$\tfrac{1}{2}\{1\pm\sqrt{1-2\{-7\pm1\pm5\pm1\}},$$

or 1, -1, -2, 3, which are seen to be correct.

Care must be exercised that the proper sign before the main radical is taken.

Urbana, Ill., October 9, 1897.

EQUATION OF PAYMENTS.

By J. A. CALDERHEAD, A. B., Professor of Mathematics, Curry University, Pittsburg, Pennsylvania.

Let it be required to find the equated time of two payments, P and P_1, due at the end of t and t_1 years respectively, and r being the rate of interest.

Represent the equated time by x when $t>t_1$.

I. BY SIMPLE INTEREST.

1st Method. The discount on P for $(t-x)$ years must equal the interest on P_1 for $(x-t_1)$ years.

$\frac{P(t-x)r}{1+(t-x)r}$=discount on P due $(t-x)$ years hence.

$P_1(x-t_1)r$=interest on P_1 for $(x-t_1)$ years.

$\therefore \frac{P(t-x)r}{1+(t-x)r}=P_1(x-t_1)r.$

$$\therefore\ x=\frac{1}{P_1 r}(P+P_1+P_1rt+P_1rt_1$$

$$\pm\sqrt{P^2+P_1^2+P_1^2r^2t^2+P_1^2r^2t_1^2+2PP_1+PP_1rt+2PP_1rt_1+P_1^2rt)} \quad \ldots\ldots\ldots (1).$$

2nd Method.

$\frac{P}{1+rt}$ = present worth of P due t years hence.

$\frac{P_1}{1+rt_1}$ = present worth of P_1 due t_1 years hence.

$\frac{P+P_1}{1+rx}$ = present worth of $P+P_1$ due x years hence.

$$\therefore \text{Suppose } \frac{P}{1+rt}+\frac{P_1}{1+rt_1}=\frac{P+P_1}{1+rx};$$

$$\text{then } x=\frac{Pt+P_1t_1+Prtt_1+P_1rtt_1}{P+P_1+Prt_1+P_1rt} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Since (2) differs from (1), the sum of the present worths of P and P_1 due in t and t_1 years respectively, at simple interest, is not equal to the present worth of $P+P_1$ due at the equated time ; hence, the second method is not correct when we compute by simple interest.

II. By Compound Interest.

1st Method.

$P\left[1-\frac{1}{(1+r)^{t-x}}\right]$ = discount on P for $(t-x)$ years.

$P_1[(1+r)^{x-t_1}-1]$ = interest on P_1 for $(x-t_1)$ years.

$$\therefore P\left[1-\frac{1}{(1+r)^{t-x}}\right]=P_1[(1+r)^{x-t_1}-1].$$

$$\therefore x=\frac{\log(P+P_1)[(1+r)^t(1+r)^{t_1}]-\log[P(1+r)^{t_1}+P_1(1+r)^t]}{\log(1+r)} \quad \ldots\ldots (3).$$

2nd Method.

$\frac{P}{(1+r)^t}$ = present worth of P due t years hence.

$\frac{P_1}{(1+r)^{t_1}}$ = present worth of P_1 due t_1 years hence.

$\frac{P+P_1}{(1+r)^x}$ = present worth of $P+P_1$ due x years hence.

$$\therefore \text{Suppose } \frac{P}{(1+r)^t}+\frac{P_1}{(1+r)^{t_1}}=\frac{P+P_1}{(1+r)^x}.$$

Then $x=\frac{\log(P+P_1)[(1+r)^t(1+r)^{t_1}-\log[P(1+r)^{t_1}+P_1(1+r)^t]}{\log(1+r)}$ ....(4).

But (4) and (3) being identical, either method may be used when compound interest is considered. The first, or correct, method by simple interest becomes very complicated when more than two payments are considered ; yet when we recall the fact that equation of payments is a subject of no practical importance, making approximate methods less desirous, it matters little how complicated the method may be if it is correct in theory.

The following method, which is found in most arithmetics is very often not much better than a good guess. A review of the solution will, at once, show the erroneousness of the method.

$P(t-x)r$=interest on P for $(t-x)$ years.

$P_1(x-t_1)r$=interest on P_1 for $(x-t_1)$ years.

$P(t-x)r=P_1(x-t_1)r$.

$\therefore x=\frac{Pt+P_1t_1}{P+P_1}$.

III. By Annual Interest.

$\frac{Pr[(t-x)+\frac{1}{2}r(t-x)(t-x-1)]}{1+r[(t-x)+\frac{1}{2}r(t-x)(t-x-1)]}$=discount on P for $(t-x)$ years.

$P_1r[(x-t_1)+\frac{1}{2}r(x-t_1)(x-t_1-1)]$=interest on P_1 for $(x-t_1)$ years.

$\therefore \frac{Pr[(t-x)+\frac{1}{2}r(t-x)(t-x-1)]}{1+r[(t-x)+\frac{1}{2}r(t-x)(t-x-1)]}=P_1r[(x-t_1)+\frac{1}{2}r(x-t_1)(x-t_1-1)]$ .....(5).

From (5) x, the equated time, can be found.

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED, A. M., (Princeton) ; Ph. D., (Johns Hopkins) ; Member of the London Mathematical Society ; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from the August-September Number.]

Proposition XXVIII. *If two straights AX, BX (produced from any-sized straight AB toward the same parts, the first under an acute angle, and the other perpendicularly) mutually approach each other ever more without any certain limit, save at their infinite production ; I say all angles (Fig. 33.) at any points L, H, D of AX, from which are dropped to the straight BX perpendiculars LK, HK, DK,*

first will all be obtuse toward the parts of the point A, secondly will be ever less, the more distant from this point A, and finally the angles more and more distant from this same point A ever more without any certain limit approach to equality with a right angle.

Demonstratur. The first part follows indeed from Corollary I to Proposition XIII. The second part however is proved thus. For the two angles together at *LK* toward the base *AB* are greater (from Corollary to Proposition XVI.) than the two internal and opposite angles together at *HK* toward the same base *AB*.

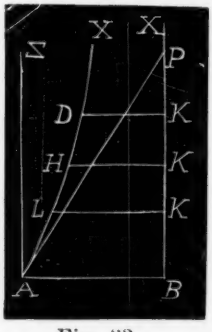


Fig. 33.

But the angles at each point *K* toward the base *AB* are equal to each other, as being right. Therefore the obtuse angle at *L* toward the base *AB* is greater than the obtuse angle at *H* toward the same base *AB*.

In like manner is shown that the aforesaid obtuse angle at *H* is greater than the obtuse angle at the point *D*.

And thus ever, proceeding toward the points *X*.

Finally the third part requires a longer disquisition. If therefore it can be done, let there be assigned (Fig. 34.) a certain angle *MNC*, than which is always greater, or anyhow not less, the excess of any of the aforesaid obtuse angles above a right angle. It follows (from Proposition XXI.) that the sides *NM*, *NC* comprehending that angle *MNC* can be so produced that the perpendicular *MC* from a certain point *M* of *MN* let fall upon *NC* may be greater (even in the hypothesis of acute angle) than any assigned finite length, as for instance the aforesaid base *AB*.

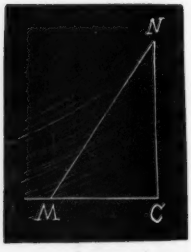


Fig. 34.

This standing ; assume in *BX* (Fig. 35.) a certain *BT* equal to *CN*, and erect from the point *T* toward *AX* the perpendicular *TS*, which obviously (from Scholion to Proposition XXIV.) meets *AX* in a certain point *S*. Then from the point *S* let fall to *AB* the perpendicular *SQ*.

This falls (because of Euclid I. 17.) toward the parts of the acute angle between the points *A* and *B*. Again, acute will be the angle *QST* in the quadrilateral *QSTB*, since the remaining three angles are right ; else (against Proposition V. and Proposition VI.) we come upon the hypothesis either of right angle or of obtuse angle.

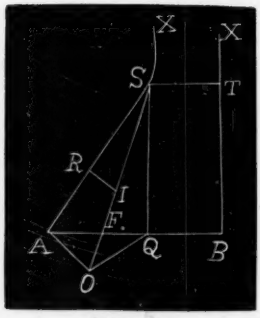


Fig. 35.

Hence the straight *SQ* will be greater (from Corollary I. to Proposition III.) than the straight *BT*,

or CN ; and again the angle ASQ will be greater than the excess by which the obtuse angle AST exceeds a right angle, and thus greater than the angle MNC. Draw therefore a certain SF cutting AQ in F and making with SA an angle equal to MNC. Then from the point A draw to SF produced the perpendicular AO. The point O falls (from Euclid I. 17.) below the point F, since the angle AFS (by Euclid I. 16.) is obtuse.

Finally, however ; since FS is greater (by Euclid I. 19.) than QS and so much greater than BT or CN, assume in FS the piece IS equal to CN, and from the point I erect to FS the perpendicular IR meeting AS in the point R.

But the point R falls between the points A and S : for if it fell on any point of AF, we would have in the same triangle (against Euclid I. 17.) two angles greater than two right angles, since the angle at the point F toward the parts of the point A has already been shown obtuse.

After so much preparation thus I conclude. Since in the quadrilateral $AOIR$ the angles at the points O and I are right, and the angle at the point A (by Euclid I. 17.) is acute because of the right angle AOS, and again the angle IRA (by Euclid I. 16.) is obtuse, since the angle RIS is right : the consequence finally is (by Corollary II. to Proposition III.) that the side AO is greater than the side IR.

But (OQ joined) the side AQ is greater (by Euclid I. 19.) than the side AQ, because of the obtuse angle at O, since the angle AOS was made right.

Therefore the straight AQ will be much greater than the straight IR, or (by Euclid I. 26.) than the straight MC, and so much greater than the straight AB, the part than the whole ; which is absurd.

Therefore it is not posssible to assign any one angle MNC, than which always is greater, or anyhow not less, the excess of each of the aforesaid obtuse angles above a right angle.

Wherefore those obtuse angles, more and more distant from this point A, ever more without any certain limit approach to equality with a right angle.

Quod erat postremo loco demonstrandum.

Corollary. But this standing, which in the last case was demonstrated, it manifestly follows that those straights AX, BX, produced infinitely will finally have, either in two distinct points, or in one same point X infinitely distant, a common perpendicular.

But again, that this common perpendicular cannot be had in two distinct points flows manifestly from this, because otherwise (by Corollary II. to Proposition XXIII.) those straights would thence begin mutually to separate, and so not meet each other at an infinite distance ; so that also (against the express supposition) they would not mutually approach each other without any certain limit ever more toward those parts.

So they must have the common perpendicular in one same point X infinitely distant.

[To be Continued.]

NEW AND OLD PROOFS OF THE PYTHAGOREAN THEOREM.

By BENJ. F. YANNEY, A. M., Mount Union College, Alliance, Ohio, and JAMES A. CALDERHEAD, B. Sc., Curry University, Pittsburg, Pennsylvania.

[Continued from June-July Number.]

XLVI. Fig. 27.

ABLN is equivalent to *ABMK* is equivalent to *ACIK*.

NLFH=ABPO is equivalent to *BEDC*.

∴ *ABFH* is equivalent to *ACIK+BEDC*.

XLVII. Fig. 27.

ABLN is equivalent to *ACIK*.

NLPO is equivalent to *STER* is equivalent to *MTERC+QFD*.

OPLH is equivalent to *REFH* is equivalent to *REFQ+MBT*.

∴ *ABFH* is equivalent to *ACIK+BEDC*.

XLVIII. Fig. 27.

AVUH is equivalent to 2*ACH* is equivalent to *ACIK*.

VBFU is equivalent to 2*CBF* is equivalent to *BEDC*.

∴ *ABFH* is equivalent to *ACIK+BEDC*.

Wipper.

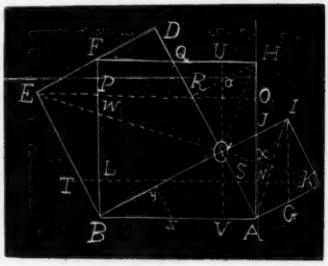


Fig. 27.

XLIX. Fig. 27.

ABWX, the half of *ABFH*, is equivalent to *ABC+CBW+CXA*.

But *ABC=BEF* (is equivalent to *BWE+AXK*).

∴ *ABWX* is equivalent to *CBE+CAK*.

∴ *ABFH* is equivalent to *ACIK+BEDC*.

L. Fig. 27.

Byz=FDQ. *AzyC=AJIK*. *ARH=BEF*. *HRQ=ACJ*.

∴ *ABFH* is equivalent to *ACIK+BEDC*.

LI. Fig. 27.

ABC=BEF. *CRa=FDQ*. *HRQ=IKG*. *HJCa* is equivalent to *IGAJ*.

∴ *ABFH* is equivalent to *ACIK+BEDC*.

That *HJCa* is equivalent to *IGAJ* is evident for the following reasons: △*ACH* is equivalent to △*ACI*, having the same base, and equal altitudes.

Hence, subtracting △*ACJ*, which is common to both, we have △*CJH* is equivalent to △*AJI*.

∴ *HJCa* is equivalent to *IGAJ*.

LII. Fig. 28.

$ARC=BEF$. $HRQ=ACJ$. $ARH=HKA$ is equivalent to $AKIJ+FDQ$.

$\therefore$ $ABFH$ is equivalent to $ACIK+BEDC$.

LIII. Fig. 28.

$AMNH$ is equivalent to $ACLH$ is equivalent to $ACIK$.

So, $MBFN$ is equivalent to $BEDC$.

$\therefore$ $ABFH$ is equivalent to $ACIK+BEDC$.

Wipper.

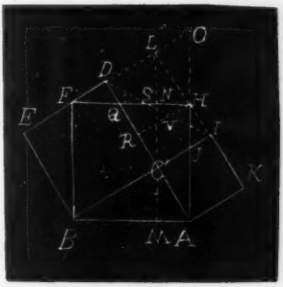


Fig. 28.

LIV. Fig. 28.

$CLOJ$ is equivalent to $CLHA$ is equivalent to $ACIK$.

$BFLC$ is equivalent to $BEDC$.

But $ABFH$ is equivalent to $BFOJ$.

$\therefore$ $ABFH$ is equivalent to $ACIK+BEDC$.

Hoffmann, 1800.

LV. Fig. 28.

$ABFH+BEF+FLH+HKA$ is equivalent to $ACIK+BEDC+ABC+CIL+CLD$.

$\therefore$ $ABFH$ is equivalent to $ACIK+BEDC$.

LVI. Fig. 28.

$ABC=BEF$. $ICD=AKH$ is equivalent to $AKIJ+FDQ$.

$SVH=SQD$, and $VHT=IJT$.

$\therefore$ By properly combining and substituting, $ABFH$ is equivalent to $ACIK+BEDC$.

LVII. Fig. 28.

$RDLH=ACIK$. $ARH=BEF$. $ABC=HFL$.

$\therefore$ $ABFH$ is equivalent to $ACIK+BEDC$.

[To be Continued.]

EUCLIDEAN GEOMETRY WITHOUT DISPUTED AXIOMS.

By G. I. HOPKINS, Instructor in Mathematics and Physics in High School, Manchester, New Hampshire.

(*a*)

PROPOSITION I. *If two straight lines in the same plane be perpendicular to the same straight line they are parallel.*

Prove by Axiom 11, and I, 27.*

*These and the subsequent numbers refer to the Book and Proposition in Todhunter's Euclid.

(*b*)

PROPOSITION II. *From or through a given point in a straight line only one perpendicular to that line can be drawn in the same plane.*

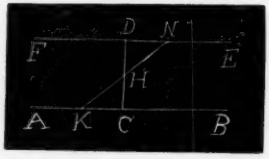
PROOF. If there could be two, there would be two unequal right angles, which is impossible by Axiom 11.

(*c*)

PROPOSITION III. *If two parallel straight lines be joined by a common perpendicular, any straight line which bisects the perpendicular and meets the two parallels is itself bisected by the perpendicular.*

Let AB be a straight line. Take any point in it as C and erect the perpendicular CD (I, XI). At D erect the perpendicular DE (I, 11) and extend it to F (Postulate 2). Then FE is parallel to AB (*a*).

Now bisect DC in H, (I, 10), take any point in AC as K and join KH, (Postulate I). On DE cut off DN equal to KC, (I, 2), and join HN, (Postulate 1). Therefore the two triangles KCH and DHN are equal to each other (I, IV). Therefore KH equals HN. Again, since the two triangles KCH and DHN are equal, the angle DHN equals the angle KHC, being homologous angles. The angles KHC and KDH are together equal to two right angles (I, 13). Therefore since the angle DHN equals the angle KHC, the angles DHN and KHD are together equal to two right angles, and therefore KH and HN form one and the same straight line (I, 14). Therefore, since K is any point in AB, any straight line which bisects the perpendicular joining two parallel straight lines is bisected by the perpendicular.

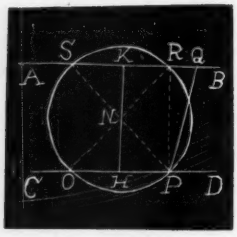


COROLLARY. If two parallel straight lines be joined by a common perpendicular, any straight line meeting the parallels and bisecting the perpendicular cuts off equal distances on the parallels on opposite sides of the perpendicular.

(*d*)

PROPOSITION IV. *If a straight line is perpendicular to one of two parallel lines it is perpendicular to the other also.*

PROOF. Let CD be a straight line. Then from any point in it as H draw HK perpendicular to CD, and in the same manner draw AB perpendicular to KH (I, 11). Then AB and CD are parallel (*a*). Take any point in one of the parallels as P in CD and suppose PQ be drawn perpendicular to CD. Then will PQ be perpendicular to AB also. For cut off $HO=HP$ (I, 2), bisect HK at N (I, 10), and draw PS and OR through N. Then $NO=NP$ (I, 4). But $SN=NP$ and $NO=NR$ (*c*). Therefore $NS=NO=NP=NR$ (Axiom 1). Therefore, similarly, $OH=HP=KR=SK$ (*c*, Corollary). With N as a center and NO as a radius describe a circle (Postulate 3). The circumference of this circle will obviously pass through the points O, P, R, and S. Draw PR. The angle NHO is greater than the an-

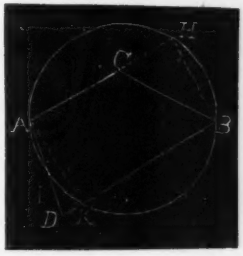


gle *NPH* (I, 16), therefore the angle *NHP* is greater than the angle *NPH*, and therefore *NP* is greater than *NH* (I, 19). Therefore the circumference of this circle will intersect the two parallel lines in the points *O*, *P*, *R*, and *S*. The angle *OPR* is a right angle (III, 31), and therefore *RP* is perpendicular to *CD*. But *QP* is by hypothesis perpendicular to *CD*, therefore *PQ* and *PR* cannot form two separate lines (*b*). Therefore *PQ*, if properly drawn must be identical with *PR*. But the angle *SRP* is a right angle (III, 31) and therefore *PQ* is perpendicular to *AB*. Q. E. D.

(*e*)

PROPOSITION V. *If the vertex of an angle subtended by the diameter of a circle is between the center and circumference, the angle is greater than a right angle; and if the vertex is without the circle the angle is less than a right angle.*

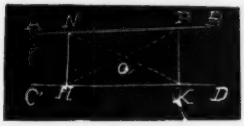
PROOF. Let *AKH* be a circle, *AB* a diameter of that circle, and let it subtend the two angles *ACB* and *D*, the vertex of the former being within, and that of the latter without, the circle. Extend *AC* to the circumference at point *H*, and join *HB* and *KA* (Postulate 1). Therefore the angles *H* and *AKB* are right angles (III, 31). Therefore the angle *ACB* is greater than angle *H* and angle *D* is less than angle *AKB* (I, 16).



(*f*)

PROPOSITION VI. *If two parallel straight lines be joined by two common perpendiculars, these two perpendiculars are equal to each other.*

PROOF. Let *AB* and *CD* be two parallel straight lines and let *NH* and *PK* be perpendicular to *CD*, then are they also perpendicular to *AB* (*d*). Join *NK* and *HP* (Postulate 1). Bisect *HP* (I, 10), then with the middle point of *HP* as a center and one-half *HP* as a radius describe a circle (Postulate 3). The circumference of this circle will obviously pass through the points *H* and *P*. It must also pass through *N* and *K*, otherwise the angles *HNP* and *HKP* would not be right angles (*e*). Again, bisect *NK* (I, 10) and with its middle point as a center and one-half *NK* as a radius describe another circle (Postulate 3). The circumference of this circle will also pass through the points *N*, *K*, *P*, and *H* for the same reason as the one above. Therefore these circumferences will coincide with one another (III, 10). Therefore there can be but one center point which being in both the lines *NK* and *HP* must be at the point of intersection *O*. Therefore the two triangles *NOH* and *POK* are equal to each other (I, 4), and therefore *NH* equals *PK*.



Q. E. D.

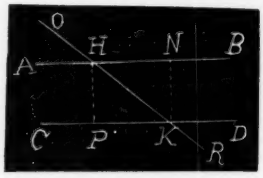
COROLLARY. The intercepts on two parallel straight lines by two common perpendiculars are equal to each other.

For, the triangles *NOP* and *HOK* are equal to each other (I, 4). Therefore *NP* is equal to *HK*, being homologous sides of two equal triangles.

(*g*)

PROPOSITION VII. *If a straight line fall on two parallel straight lines, it makes the alternate angles equal to one another, etc.* (I, 29).

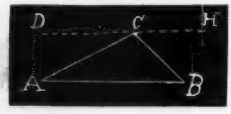
PROOF. Let the straight line *OR* fall on the two parallel straight lines *AB* and *CD*, meeting them in points *H* and *K* respectively. Then the angles *BHK* and *CKH* shall be equal to one another.



From *H* draw *HP* perpendicular to *CD*, and from *K* draw *KN* perpendicular to *AB* (I, 12). Then *HP* is also perpendicular to *AB* and *KN* is also perpendicular to *CD* (*d*). Therefore *HP* equals *NK* (*f*), and *HN* equals *PK* (*f* Corollary). Therefore the two triangles *HPK* and *HNK* are equal to each other (I, 8), and therefore the angle *NHK* equals the angle *HKP*, being homologous angles of two equal triangles. Q. E. D.

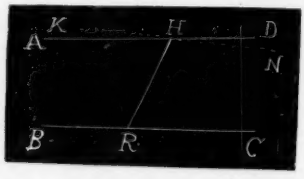
PROPOSITION VIII. *The sum of the angles of every plane triangle is equal to two right angles.*

PROOF. Let *ABC* be any plane triangle, then the sum of the angles *A*, *B*, and *C* is equal to two right angles. Through one of its vertices as *C* draw *DH* parallel to *AB* (I, 31). Then the angles *A* and *DCA* are equal to one another (*g*), as are also the angles *B* and *HCB* for the same reason. But the sum of the angles *DCA*, *ACB*, and *BCH* is equal to two right angles (I, 13). Therefore the sum of the angles *A*, *B*, and *ACB* must equal two right angles. Q. E. D.



PROPOSITION IX. *Through a given point without a given straight line only one line can be drawn parallel to the given line.*

PROOF. Let *BC* be a straight line and *H* a point without. Draw *AD* through *H* parallel to *BC* (I, 31). Then no other line can be drawn through *H* parallel to *BC*. If possible suppose *KN* drawn through *H* parallel to *BC*. Then since the angles *KHR* and *AHR* are each equal to the angle *HRC* (*g*), they are equal to each other (Axiom 1), a part to the whole which is impossible. Therefore *KN* cannot be parallel to *BC*. Q. E. D.

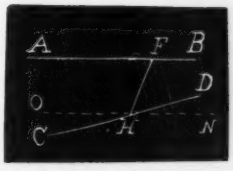


PROPOSITION X. *If a straight line fall on two parallel straight lines, the sum of the two interior angles on the same side of that line shall be equal to two right angles.*

PROOF. Let the straight line *ON* fall on the two parallel straight lines *AB* and *CD*. Then the sum of the two angles *AKH* and *CHK* is equal to two right angles. For, the sum of the two angles *CHK* and *KHD* is equal to two right angles (I, 13) and the angle *AKH* equals the angle *KHD* (*g*). Therefore, substituting the latter for the former we have the sum of the two angles *AKH* and *CHK* equal to two right angles. Q. E. D.

PROPOSITION XI. *If a straight line meet two straight lines so as to make the two interior angles on the same side of it taken together less than two right angles, these straight lines, being continually produced, shall at length meet on that side on which are the angles which are less than two right angles.* Euclid, Axiom 12.

PROOF. Let the straight line FH meet the two straight lines AB and CD, making the two angles BFH and FHD together less than two right angles, then AB and CD shall meet, if continually produced, on that side of FH towards B and D. Since the angles BFH and AFH are together equal to two right angles, they must be greater than the sum of the two angles BFH and FHD. Therefore, the angle AFH must be greater than the angle FHD. Hence, draw the line ON through H making the angle FHN equal to the angle AFH (I, 23). Then ON is parallel to AB (I, 27). Therefore CD cannot be parallel to AB (i), and therefore CD and AB must meet if sufficiently produced. Since the sum of the angles AFH and FHO equals two right angles (j), the sum of the angles AFH and FHC must be greater than two right angles. Therefore AB and CD cannot meet on that side of FH toward A and C for then we should have a triangle the sum of whose angles would be greater than two right angles which is impossible by (h). Therefore they must meet on that side of FH toward B and D. Q. E. D.



ZERO, INFINITESIMALS, INFINITY, AND THE FUNDAMENTAL SYMBOL OF INDETERMINATION.

By GEORGE LILLEY, Ph. D., Professor of Mathematics, State University, Washington.

The following is an outline of the method I use in explaining to the student in algebra how zero is used as a multiplier and a divisor, and how infinitesimals and infinity are used as divisors; also, interpretations of the results obtained by their use.

If we multiply a by a number that decreases by 1 each time beginning with any number, as $+4$, and continue the multiplication until -4 is reached, each product will decrease by a. Thus,

a	a	a	a	a	a	a	a
$+4$	$+3$	$+1$	*zero*	-1	-2	-3	-4
$+4a$	$+3a$	$+a$,	zero,	$-a$	$-2a$	$-3a$	$-4a$,

where zero is a constant number and obtained by subtracting *any* number from itself, as, $a-a=⓪$, ⓪ representing *absolute* zero.

Evidently a multiplied by zero is one a less than a multiplied by $+1$, or

$a \times ⓪ = ⓪$; also, a multiplied by -1 is one a less than a multiplied by zero, or $a \times -1 = -a$. Similarly $a \times -2 = -2a$, $a \times -3 = -3a$, etc.

Hence, *If a constant number be multiplied by zero, the product is zero.*

Division may be defined as the process of finding how many times the divisor can be subtracted from the dividend and leave zero.

Dividing 12 by a number that decreases by unity each time, beginning with $+3$, we have

$$\frac{12}{+3}=4,\ \frac{12}{+2}=6,\ \frac{12}{+1}=12,\ \left(\frac{12}{\text{zero}}\right),\ \frac{12}{-1}=-12,\ \frac{12}{-2}=-6,\ \frac{12}{-3}=-4\ ;\ \text{etc.}$$

The quotient 4 means that only 3 times $+4$ can be subtracted from 12 and leave zero ; and so on for the other quotients.

Since the divisor decreases by unity, the divisor one less than 1 is zero. The divisors less than zero are -1, -2, -3, etc., respectively. Then, the quotient, when zero becomes the divisor, must be between the quotients given by taking $+1$ and -1 as divisors, or between $+12$ and -12.

Then $\frac{12}{\text{zero}}$ or $\frac{12}{⓪}=\ominus$, where $\ominus$ represents no number of times. That is, there is *no number of times zero* that the divisor, zero, can be subtracted from 12 and *leave absolutely nothing.*

Since negative numbers are less than zero, ⓪ is not the least divisor of 12, or of any other number. If $\frac{12}{⓪}$=infinity, or the *largest possible number*, 12 divided by -1 can not give -12 for a quotient. If $\frac{12}{-1}=-12$, $\frac{12}{\text{zero}}$ or $\frac{12}{⓪}$ can not give infinity for a quotient, for the divisor, -1, is one less than the divisor ⓪.

Hence, in general, $\frac{a}{⓪}=\ominus$.

For the quotient, $\ominus$, means that there is no number of times zero that the divisor, ⓪, can be subtracted from a and leave zero.

Hence, *If a constant number be divided by zero, the quotient is no number of times.*

It is a consequence of confounding the 0, arising from dividing a by *infinity*, with the absolute zero, that so much confusion has been created in the discussions on this subject. All absolute zeros are constants. The other 0's, used in these discussions, are infinitesimals and variables, and may be less than ⓪.

Since an infinitesimal can be subtracted from a an infinite number of times and leave zero ; therefore, $\frac{a}{◎}=\infty$, where ◎ represents an infinitesimal.

That is, *If a constant number be divided by an infinitesimal, the quotient is infinity.*

Suppose, for illustration, we divide a by a number that diminishes one-

tenth each time, beginning with one ; we will have the series

$$\frac{a}{1}=a,\ \frac{a}{\frac{1}{10}}=10a,\ \frac{a}{\frac{1}{100}}=100a,\ \frac{a}{\frac{1}{1000}}=1000a,\ \frac{a}{\frac{1}{10000}}=10000a,\ \ldots\ldots$$

Evidently, by continuing the series indefinitely, the divisor becomes *less* than any assignable number however small, and the value of the quotient increases without limit and becomes *greater* than any assignable number however great.

Hence, *If a constant number be divided by a decreasing variable, as the variable becomes too small to be expressed, the quotient becomes too large to be expressed.*

Since infinity can be subtracted from a the infinitesimal part of once and leave zero ; therefore, $\frac{a}{\infty}=\circledcirc$.

That is, *If a constant number be divided by infinity, the quotient is infinitesimal.*

Suppose the divisor, in the above illustration, increases each time, beginning with 1 ; we will have the series

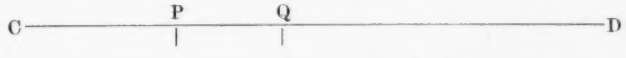
$$\frac{a}{1}=a,\ \frac{a}{10}=.1a,\ \frac{a}{100}=.01a,\ \frac{a}{1000}=.001a,\ \frac{a}{10000}=.0001a,\ \ldots\ldots$$

Evidently, by continuing the series indefinitely, the divisor becomes greater than any assignable number, however great, and the value of the quotient decreases without limit and becomes *less* than any assignable number, however small.

Hence, *If a constant number be divided by an increasing variable, as the variable becomes too great to be expressed, the quotient becomes too small to be expressed.*

This subject is also illustrated in interpreting the results obtained by assigning different values for the rates of travel and the distance apart, in Clairaut's problem, of the Couriers.

"Two Couriers, A and B, travel in the same direction, CD, at the rates m and n miles an hour, respectively. If at any time, say 12 o'clock, A is at P, and B is at Q, a miles from P, at what time and at what place are they together ?"



Let t=the number of hours traveled, after 12 o'clock, to the place where A overtakes B, and d=the number of miles travelled by A in t hours ; or the number of miles from P to the place where A overtakes B.

Since the number of miles travelled by each, after 12 o'clock, equals the rate multiplied by the number of hours, we have

$$d=mt,\text{ and } d-a=nt.$$

Solving these equations, we have

$$t=\frac{a}{m-n}, \quad d=\frac{am}{m-n}.$$

We will now examine these values under different conditions.

1. If $m>n$.

This condition makes the values of t and d *positive*. That is, the Couriers are together *after* 12 o'clock, and at some place to the *right* of P.

If $m<n$.

This condition makes the values of t and d *negative*. That is, the Couriers are together *before* 12 o'clock, and to the *left* of P. This interpretation corresponds with the conditions made. For, if m is less than n, A travels more slowly than B, and it follows that they must have been together before 12 o'clock, and before they could have advanced as far as P.

3. If $m=n$, or $m-n=$zero.

Then $t=\frac{a}{⓪}$, and $d=\frac{am}{⓪}$.

As there is no number of times zero that subtracted from a leaves zero, there is no number of hours when they have been or will be together. Furthermore, as every number of times zero subtracted from a leaves a; that is, $a-v\times ⓪=a$, where v represents any number whatever, they are always the same distance apart.

Hence, *A result in the form* $\frac{a}{⓪}$ *indicates that the problem is impossible.*

This interpretation corresponds with the supposition made. For, if m is equal to n, the Couriers travel at the same rate, and since they were a miles apart at 12 o'clock, it is evident they never could have been, and never will be, together.

4. If $a=$zero, and $m>n$, or $m<n$.

Then $t=\frac{⓪}{m-n}$, and $d=\frac{⓪}{m-n}$.

They are together at the start, as shown by $a=$zero; but, as there is no number of times $m-n$ that subtracted from zero, will leave zero, they can never be together again.

Furthermore, the longer the time, the greater or less will $m-n$ be; hence, they will be constantly diverging. For example,

In 1 hour, zero$-(m-n)=n-m$, distance apart;

In 2 hours, zero$-2(m-n)=2n-2m$, distance apart;

In 3 hours, zero$-3(m-n)=3n-3m$, distance apart, etc.

Hence, $t=\frac{⓪}{m-n}$ indicates that they will be together in zero hours after 12 o'clock, but never after or before. For zero$-⓪\times(m-n)=$zero, is the only value for t that will satisfy the conditions.

Similarly, $d=\frac{①}{m-n}$ means no distance from P, and shows that they were placed together by the conditions of the problem, a=zero, but for *all other time the problem is impossible.*

5. If a=zero, and $m=n$.

Then $t=\frac{①}{①}$, and $d=\frac{①}{①}$.

As any number of zeros subtracted from zero gives zero; that is, zero$-v\times①$=zero, where v represents any number whatever, they are together at all times; for t=*any number.*

Hence, $t=\frac{①}{①}$ means all conceivable times, and $d=\frac{①}{①}$ means all conceivable distances, and are indeterminate, not being one, but every value.

Therefore, *A result* $\frac{①}{①}$ *indicates that the problem is indeterminate.*

The form, $\frac{①}{①}$, is a *symbol of indetermination*, and does not indicate that no solution can be found, but that too many can be determined. The indetermination consists in the fact that any one of the infinite solutions will answer just as well as any other.

January 11, 1897.

EDITORIALS.

This issue of the MONTHLY was delayed on account of securing sorts for Dr. Lovett's article.

Prof. E. L. Brown, formerly of the Capital University, Columbus, Ohio, is now a member of the Faculty of the Department of Mathematics of the Colorado State University.

The articles on "Euclidean Geometry Without Disputed Axioms," and "Zero, Infinitesimals, Infinity, and the Fundamental Symbol of Indetermination," are published at the request of the authors. They invite criticism on their respective articles, and, if there is any defect in the reasoning by which they arrived at their conclusions, they desire to have the same pointed out.

BOOKS AND PERIODICALS.

A Text-Book of Light. With Numerous Diagrams and Examples. By R. Wallace Stewart, D. Sc., London, Author of "An Elementary Text-Book of Heat and Light," "An Elementary Text-Book of Magnetism and Electricity," etc.

Third Edition. 8vo Cloth, 208 pages. Price, 3*s.* 6*d.* London : W. B. Cleve University Correspondence College Press. New York : Hinds & Noble, 4 Cooper Institute.

This little treatise on Light is clearly, neatly, and accurately written. The author as a teacher and writer needs no introduction. His works all bear evidence of a master of the subject under consideration. This book in the hands of the student will enable him to read with interest and profit the investigations in this most fascinating phenomenon of nature. B. F. F.

On the Transitive Substitution Groups that are Simply Isomorphic to the Symmetric or Alternating Group of Degree Six. By Dr. G. A. Miller.

This is a reprint of a paper read before the American Philosophical Society May 7, 1897, and published in the proceedings of that Society. B. F. F.

New Principles of Geometry with Complete Theory of Parallels. By Nicoláï Ivánovish Lobachévski. Translated from the Russian by Dr. George Bruce Halsted. Volume fifth of the Neomonic Series.

The publication of the translation of this little pamphlet of 26 pages promised by Dr. Halsted at the Mathematical Congress of the World's Columbian Exposition was delayed for a personal visit to Kazan, the home of Lobachévski, and Maros Vásárhely, the home of Bolyai.

In this pamphlet is some interesting matter for the student of Non-Euclidean Geometry, and it should be read by every teacher of geometry. Several of our readers have written to us saying that they could see no sense in the Non-Euclidean Geometry. We say to these, read this little pamphlet, then with the light you get from it turn back and read the first and all subsequent articles of Dr. Halsted's which have appeared in the MONTHLY during the last four years, then turn to other sources for information on the subject. After having done this, you will find that there is a Non-Euclidean Geometry, that its argument is as rigorous as the Euclidean, and that its deductions are equally interesting. B. F. F.

A Text-Book of Physics. Largely Experimental, including the Harvard College "Descriptive List of Elementary Exercises in Physics." By Edwin H. Hall, Ph. D., Professor of Physics in Harvard College, and Joseph Y. Bergen, A. M., Instructor in the Harvard Summer School of Physics, and Junior Master in English High School, Boston. Revised and Enlarged. 8vo Cloth. 596 pages. New York : Henry Holt & Co.

This book needs no introduction to the public. The first edition has proved its usefulness in all experimental courses in Physics. The second edition is even an improvement over the first. B. F. F.

Ordinary Differential Equations. An Elementary Text-Book with an Introduction to Lie's Theory of the Group of One Parameter. By James Morris Page, Ph. D., University of Leipzig, Fellow by Courtesy Johns Hopkins University, Adjunct Professor of Pure Mathematics in University of Virginia. 8vo Cloth. 226 pages. New York and London : The Macmillan Co.

This is the best elementary exposition of the subject of Ordinary Differential Equations with which we are acquainted. It differs from the older text-books upon the subject in one important respect, namely, in the method of treatment. Instead of giving theories of integration for certain classes of Differential Equations, as for instance, the Homogen-

eous or Linear Differential Equations as is done in the older works on the subject, the author has followed the method of Professor Lie. In 1870, Lie showed that it is possible to subordinate all the older methods or theories of integration to a general method. By the method of Lie it is possible to derive all of the older theories from a common source and at the same time build a broader foundation for the general theory of Differential Equations. The simple, elegant, and clear presentation of the subject in this work makes it possible for a student who has ambition and a fair knowledge of Analytical Geometry and Calculus to master this book without an instructor. B. F. F.

On the Primitive Substitution Groups of Degree Fifteen. By Dr. G. A. Miller. Pamphlet, 12 pages.

This paper is an extract from the Proceedings of the London Mathematical Society, Vol. XXVIII., and is along Dr. Miller's favorite line of investigation. B. F. F.

Higher Arithmetic. By Wooster Woodruff Beman, Professor of Mathematics in the University of Michigan, and David Eugene Smith, Professor of Mathematics in the Michigan State Normal School. 12mo. Cloth and Leather Back. 194 pages. Price, 80 cents. Chicago : Ginn & Co.

Among the many valuable features of this work are the elimination of the traditional problems which have become the common property of nearly every arithmetic published during the last quarter century; the introduction, instead of the traditional problems, of simple problems arising in the study of elementary physics, as, for example, easy problems in Electrical Measurements, problems coming under the application of Boyle's Law, the law of Falling Bodies, Specific Gravity, etc.; the treatment of the Metric System in the first part of the book (page 59) and the frequent use made of it in the subsequent part; the introduction of the common graphic methods of representing statistics; and the complete omission of rules. The entire omission of rules is a very common feature of the arithmetics which are published at the present time and of those which have been published during the last three or four years. It is my belief that all rules that can not be established easily by the deductive method of reasoning should be set in *good print* in the arithmetics. This is especially the case with the rules in Mensuration. The student of arithmetic is in general not competent to follow the argument which establishes the rule for finding the area of a triangle when three sides are given. Yet it is better for the student that he commit this rule to memory, though he does not know how it is established, than to be ignorant of its existence and the means by which the area of triangles are computed when the sides are given. It seems to me that any arithmetic treating the subject of mensuration ought to give the rules for finding the surface and volume of the three round bodies, the area of parallelograms, circles, and triangles, the triangles having the base and altitude given or the three sides. To these might be added the rules for finding the surface and volume of prisms and pyramids. Aside from these I heartily believe in the omission of rules. The work before us does not omit the consideration of the most of the above geometrical magnitudes, but the rules are not expressly stated. This work of Professors Beman and Smith is, however, one that we can very cheerfully recommend. B. F. F.

A Brief Introduction to the Infinitesimal Calculus. Designed Especially to Aid in Reading Mathematical Economics and Statics. By Irving Fisher, Ph. D., Assistant Professor of Political Science in Yale University, Co-author of Phillips and Fisher's Elements of Geometry. 12mo. Cloth. 84 pages. Price, 75 cents. New York and London : The Macmillan Co.

This little work on the Calculus will be received with joy by a great army of stud-

ents, teachers, and professors, who have lacked the time and courage to attack some of the more exhaustive works on the subject yet felt the need of a knowledge of the Calculus in order to enable them to read with intelligence the highest authorities on economic as well as other subjects. Dr. Fisher has prepared this little work with a special view of the needs of this class of students. Any one with a clear mind can very easily read and understand every sentence in this book. There is no metaphysical speculation nor obscure statements made in establishing its first principles.

In considering the formula $s=\frac{1}{2}gt^2$, where s=space a body falls under the influence of gravity in the time t, he says, pages 2 and 3, "Since the above formula holds true of *all* points, it holds true now, when the time is $t+\triangle t$ and the distance $s+\triangle s$. That is $s+\triangle s=16(t+\triangle t)^2$. This gives $s+\triangle s=16t^2+32t.\triangle t+16\triangle t^2$. But $s=16t^2$. Subtracting, we have

$$\triangle s=32t.\triangle t+16(\triangle t)^2,$$

$$\text{whence } \frac{\triangle s}{\triangle t}=32t+16\triangle t \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

This is the *average* velocity during the small interval $\triangle t$.

Thus, if $\triangle t=\frac{1}{2}$ second and t be five seconds, the average speed of the body during that half second (viz., the one beginning 5 seconds from the rest) is $32\times 5+19\times\frac{1}{2}$, or 168 per second. If we take $\frac{1}{160}$ of a second instead of $\frac{1}{2}$, we have $32\times 5+16\times\frac{1}{160}$, or 168.1 feet per second.

The speed at the very instant of completing the 5th second is obtained by putting $\triangle t=0$, which gives 160 as the instantaneous speed.

Now when $\triangle t=0$, we call it dt, because 0 would not remind us of the kind of quantity which vanished, whereas dt does suggest t, the magnitude which vanished. When $\triangle t$ becomes 0, or dt, $\triangle s$ evidently becomes zero too, for a body can not go any distance in no time. This zero we call ds. Equation (1) therefore becomes at the limit

$$\frac{ds}{dt}=32t+[16]dt,$$

$$\text{or } \frac{0}{0}=32t+0, \text{ which may be written}$$

$$\frac{ds}{dt}=32t \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

for we can neglect the zeros on the right, but not those on the left (the ratio of two zeros does not reduce to zero).

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It may be objected to the reasoning in the last article that $\frac{ds}{dt}$ or $\frac{0}{0}$ is indeterminate. This is true. $\frac{0}{0}$ is equal to 2, or 19, or 1, or any number we we please. *But the limit* $\frac{\triangle s}{\triangle t}$ *is not indeterminate.* We thus use $\frac{ds}{dt}$ in two distinct senses, viz : $\lim\frac{\triangle s}{\triangle t}$ and $\frac{\lim\triangle s}{\lim\triangle t}$.

The first is determinate, the second is indeterminate, though for that very reason it may always be put equal to the first. Only the first, or $\lim\frac{\triangle s}{\triangle t}$ is important. This is the ultimate ratio of two vanishing quantities."

Excepting the statement in the next to the last sentence, viz., that $\lim \frac{\triangle s}{\triangle t}$ is determinate and $\frac{\lim \triangle s}{\lim \triangle t}$ indeterminate, we claim that Dr. Fisher has established the fundamental principles of the Differential Calculus in a simple, rigorous, and logical manner. By the principle of Limits, the $\lim \frac{\triangle s}{\triangle t} = \frac{\lim \triangle s}{\lim \triangle t}$, that is to say, the limit of the quotient of two variables equals the quotient of their limits. Then if one is determinate the other is determinate, or if one is indeterminate the other is indeterminate. They are, however, both determinate. The statement that dt is used instead of 0 to preserve the trace of the quantity that vanished will be considered by many mathematicians as the rankest sort of mathematical heresy, the reanimating of Berkeley's "ghost of departed quantities." But here too Dr. Fisher's position is absolutely impregnable, for, since $\frac{0}{0}$ is, *per se*, indeterminate, but determinate by the equation by which, in every case, it is defined, it may be replaced by the ratio of any two quantities which preserves the ratio that defines $\frac{0}{0}$. So in the case above, $\frac{0}{0}$ can be replaced by $\frac{ds}{dt}$ or $\frac{y}{x}$ or the quotient of any other two quantities which preserve the ratio $32t$. But $\frac{0}{0}$ is replaced by $\frac{ds}{dt}$ to preserve the trace of the quantities which vanished, and ds and dt can represent large or small parts of s and t. There are in general three possible ways by which the ratio $32t$ can be preserved by $\frac{ds}{dt}$. First, ds being considered a constant, and dt a variable; second, ds being a variable, and dt a constant; third, $\frac{ds}{dt}$ both being variables. Each of these three ways of viewing $\frac{ds}{dt}$ is used in the Calculus. This method of exposition is used in my classes with the result that students are enabled to use the Calculus, not as a machine by which to grind out problems, but as an instrument of research.

The American Monthly Review of Reviews. An International Illustrated Monthly Magazine. Edited by Dr. Albert Shaw. Price, $2.50 per year in advance. Single Number, 25 cents. The American Monthly Review of Reviews Co., 13 Astor Place, New York City.

The American Monthly Review of Reviews for October has several articles of unusual interest to women readers. Miss Frances Willard tells the story of the World's W. C. T. U. movement; Mrs. Ellen M. Henrotin, president of the General Federation of Women's Clubs, outlines the benefits of those organizations; Mrs. Sheldon Amos, of England, writes of a London Women's Club, and Miss Mary Taylor Blauvelt contributes an enlightening article on the opportunities for women at the English universities. B. F. F.

The Arena. An Illustrated Monthly Magazine. Edited by John Clark Ridpath, LL. D. Price, $2.50 per year in advance. Single Number, 25 cents. Boston: The Arena Co.

The Arena for October continues the battle for reform. The number is especially interesting and in some parts brilliant; it is in all parts aggressive and courageous. Hon. Charles A. Towne's article, "The New Ostracism," is in the author's best vein of critical analysis. In the course of the discussion he attacks with great vigor the plutocratic interference with professors in colleges and universities. Herman E. Taubeneck continues his cogent statistical attack on consecrated wealth. Judge Walter Clark sends out a powerful plea for the establishment of public rights over semi-public interests and institutions. The Editor of *The Arena* continues with unabated vigor his onslaught on the organized forces of plutocracy. His article, "Prosperity: the Sham and the Reality," is one of his strongest and best. Dr. Ridpath's exposition of the bottom purposes and methods of the money power is as caustic as it is true. Mary Platt Parmelee's article on the Political Philosophy of the Father of American Democracy is an original and forceful argument for popular liberties. B. O. Flower is again at his best pace in "The Latest Social Vision," in which he discusses the merits of Bellamy's "Equality." Perhaps the most radical and defiant article in the number is "The Dead Hand in the Church," by Rev. Clarence Lathbury, in which he attacks with destructive criticism the domination of the dead past over the living present in the church. "Hypnotism in its Scientific and Forensic Aspects" is the subject of an interesting and useful article by Marion L. Dawson. "Suicide: Is It Worth While?" is the caption of Charles B. Newcomb's startling study of one of the most interesting and painful themes of the age. The "Plaza of the Poets" is rich with the contributions of Ironquill, Junius Hempstead, Clinton Scollard, Reubie Carpenter, and Helena M. Richardson; while "The Editor's Evening" sparkles with its usual gems of social and poetical philosophy. Under "Book Reviews" the charming poems of Madison Cawein are set forth with merited commendation. B. F. F.

The Open Court. A Monthly Magazine Devoted to the Science of Religion, and the Religion of Science, and the Extension of the Religious Parliament Idea. Edited by Dr. Paul Carus. T. J. McCormack, Assistant Editor, and E. C. Hegeler and Mary Carus, Associate Editors. Price, $1.00 per year in advance. Single Copies, 10 cents. Chicago and London: The Open Court Publishing Co.

The following is the table of contents of the November number: "An Introduction to the Study of Ethnological Jurisprudence," by the Late Justice Albert Hermann Post, Bremen, Germany; "History of the People of Israel from the Beginning to the Destruction of Jerusalem by the Chaldeans," by C. H. Cornill, Professor of Theology in the University of Koenigsberg; "The Religion of Science; the Worship of Beneficence," by James Odgers Knutsford, England; "Death in Religious Art," by the Editor; "Vivisection from an Ethical Point of View: A Controversy," by Prof. Henry C. Mercer, and others; "Leonhard Euler," a biographical sketch by T. J. McCormack; "The Sacred Books of the Buddhists," by Albert J. Edmunds; "Brief Notes on some Recent French Philosophical Works;" Book Reviews, and Notes. Among the book reviews is a just estimate or criticism of "Finkel's Mathematical Solution Book;" the review contains about a page and a half, and is written by Assistant Editor T. J. McCormack. B. F. F.

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